

# PROBABILISTIC INTERPRETATION OF THE CALDERÓN PROBLEM

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ABSTRACT. In this paper, we use the theory of symmetric Dirichlet forms to give a probabilistic interpretation of Calderón's inverse conductivity problem in terms of reflecting diffusion processes and their corresponding boundary trace processes.

## 1. INTRODUCTION

Electrical impedance tomography (EIT) aims to reconstruct the unknown conductivity  $\kappa$  in the conductivity equation

$$(1) \quad \nabla \cdot (\kappa \nabla u) = 0 \quad \text{in } D$$

from current and voltage measurements on the boundary of the domain  $D$ . This inverse conductivity problem is known to be severely ill-posed, that is, its solution is extremely sensitive with respect to measurement and modeling errors. The uniqueness question, first proposed by Calderón has been studied extensively. And while it has been answered affirmatively for isotropic conductivities in the two-dimensional case by Astala and Päiväranta [3], it is still unsettled, at least in full generality, in higher dimensions. In the previous work [22] the authors have developed a probabilistic interpretation of the forward problem in form of a Feynman-Kac formula using reflecting diffusion processes. In this work, we extend this probabilistic interpretation to Calderón's inverse problem. More precisely, we study the *time-changed process*  $\hat{X}$  on  $\partial D$  of the reflecting diffusion process  $X$  with respect to the so-called *local time on the boundary*. For the special case of the reflecting Brownian motion on the planar unit disk, the Beurling-Deny decomposition of this so-called *boundary process* is given by the well-known *Douglas integral*

$$\int_D |\nabla \mathcal{H}\phi(x)|^2 dx = \int_{\partial D \times \partial D \setminus \delta} (\phi(\xi) - \phi(\eta))^2 (4\pi(1 - \cos(\xi - \eta)))^{-1} d\xi d\eta,$$

where  $\mathcal{H}\phi$  denotes the harmonic function on  $D$  with Dirichlet boundary value  $\phi$ . As a consequence,  $\hat{X}$  is of pure jump type and its jumping mechanism, the *Lévy system*, is governed by the *Feller kernel*  $(4\pi(1 - \cos(\xi - \eta)))^{-1}$ . In this special case,  $\hat{X}$  is the *symmetric Cauchy process* on the unit circle, which leads somewhat naturally to the following *probabilistic inverse problem*: *Is the reflecting Brownian motion the unique reflecting diffusion process on the planar unit disk, whose boundary process is the symmetric Cauchy process on the unit circle?* Clearly, this question can be answered affirmatively for isotropic conductivities, however, the authors are not aware of a *probabilistic* proof.

The major goal of this work is to generalize the considerations from above in order to formulate probabilistic inverse problems which are equivalent to Calderón's problem.

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We prove that the boundary process is of pure jump type and that its jumping measure is governed by the Lévy system of the boundary process. Moreover, we show that the latter is completely determined by the transition kernel density of the corresponding *absorbing diffusion process* on  $D$ . We give explicit descriptions of both the Lévy system as well as the infinitesimal generator of the boundary process. This generalizes results by Hsu [17] for the reflecting Brownian motion and enables the formulation of Calderón's problem in the form of equivalent probabilistic inverse problems in terms of the *excursion law* of the reflecting diffusion process and the Lévy system of the boundary process, respectively. We would like to stress the fact that in contrast to the reflecting Brownian motion, the reflecting diffusion process we study in this work are in general not in the class of solutions to stochastic differential equations. In particular, we can not rely on Itô calculus.

The rest of this paper is structured as follows: We start in Section 2 by introducing our notation. In Section 3, we recall both the forward problem of EIT as well as its probabilistic interpretation from [22]. Then, in Section 4, we present two equivalent methods to define the boundary process of a reflecting diffusion. The first one uses the so-called *trace Dirichlet form* and its potential theory, whereas the second one is purely probabilistic using time change with respect to the boundary local time. Moreover, we prove that the Dirichlet-to-Neumann map is the *infinitesimal generator* of the boundary process. Finally, we study a Lévy system of the boundary process and the *excursions* of  $X$  which leads to the formulation of three equivalent probabilistic inverse problems in Section 5.

## 2. NOTATION

Let  $D$  denote a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d \geq 2$ , with connected complement and Lipschitz parameters  $(r_D, c_D)$ , i.e., for every  $x \in \partial D$  we have after rotation and translation that  $\partial D \cap B(x, r_D)$  is the graph of a Lipschitz function in the first  $d - 1$  coordinates with Lipschitz constant no larger than  $c_D$  and  $D \cap B(x, r_D)$  lies above the graph of this function. Moreover, we set  $\mathbb{R}_-^d := \{x \in \mathbb{R}^d : x \cdot \nu < 0\}$ , with  $\nu = e_d$  the outward unit normal on  $\mathbb{R}^{d-1}$ , where we identify the boundary of  $\mathbb{R}_-^d$  with  $\mathbb{R}^{d-1}$ , with straightforward abuse of notation.

For Lipschitz domains, there exists a unique outward unit normal vector  $\nu$  a.e. on  $\partial D$  so that the real Lebesgue spaces  $L^p(D)$  and  $L^p(\partial D)$  can be defined in the standard manner with the usual  $L^p$  norms  $\|\cdot\|_p$ ,  $p = 1, 2, \infty$ . The standard  $L^2$  inner-products are denoted by  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_{\partial D}$ , respectively. The  $d$ -dimensional Lebesgue measure is denoted by  $m$ , the  $(d - 1)$ -dimensional Lebesgue surface measure is denoted by  $\sigma$  and  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{R}^d$ .

All functions in this work will be real-valued and derivatives are understood in distributional sense. We use a diamond subscript to denote subspaces of the standard Sobolev spaces containing functions with vanishing mean and interpret integrals over  $\partial D$  as dual evaluations with a constant function, if necessary. For example, we will frequently use the spaces

$$H_\diamond^{\pm 1/2}(\partial D) := \left\{ \phi \in H^{\pm 1/2}(\partial D) : \langle \phi, 1 \rangle_{\partial D} = 0 \right\}$$

and

$$H_\diamond^1(D) := \left\{ \phi \in H^1(D) : \langle \phi, 1 \rangle = 0 \right\}.$$

Moreover, we will frequently assume that  $\partial D$  is partitioned into two disjoint parts,  $\partial_1 D$  and  $\partial_2 D$ . We denote by  $H_0^1(D \cup \partial_1 D)$  the closure of  $C_c^\infty(D \cup \partial_1 D)$ , the linear

subspace of  $C^\infty(\overline{D})$  consisting of functions  $\phi$  such that  $\text{supp}(\phi)$  is a compact subset of  $D \cup \partial_1 D$ , in  $H^1(D)$ . For the reason of notational compactness, we use the Iverson brackets: Let  $S$  be a mathematical statement, then

$$[S] = \begin{cases} 1, & \text{if } S \text{ is true} \\ 0, & \text{otherwise.} \end{cases}$$

We also use the Iverson brackets  $[x \in B]$  to denote the indicator function of a set  $B$ , which we abbreviate by  $[B]$  if there is no danger of confusion.

In what follows, all unimportant constants are denoted  $c$ , sometimes with additional subscripts, and they may vary from line to line.

### 3. THE FORWARD PROBLEM AND ITS PROBABILISTIC INTERPRETATION

We assume that the, possibly anisotropic, conductivity is defined by a symmetric, matrix-valued function  $\kappa : D \rightarrow \mathbb{R}^{d \times d}$  with components in  $L^\infty(D)$  such that  $\kappa$  is uniformly bounded and uniformly elliptic, i.e., there exists some constant  $c > 0$  such that

$$(2) \quad c^{-1}|\xi|^2 \leq \xi \cdot \kappa(x)\xi \leq c|\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^d \text{ and a.e. } x \in D.$$

The forward problem of electrical impedance tomography can be described by different measurement models. In the so-called *continuum model*, the conductivity equation (1) is equipped with a co-normal boundary condition

$$(3) \quad \partial_{\kappa\nu} u := \kappa\nu \cdot \nabla u|_{\partial D} = f \quad \text{on } \partial D,$$

where  $f$  is a measurable function modeling the signed density of the outgoing current. The boundary value problem (1), (3) has a solution if and only if

$$(4) \quad \langle f, 1 \rangle_{\partial D} = 0.$$

Physically speaking, this means that the current must be conserved. Given an appropriate function  $f$ , the solution to (1), (3) is unique up to an additive constant, which physically corresponds to the choice of the ground level of the potential. If  $f \in H_\diamond^{-1/2}$ , then there exists a unique equivalence class of functions  $u \in H^1(D)/\mathbb{R}$  that satisfies the weak formulation of the boundary value problem

$$\int_D \kappa \nabla u \cdot \nabla v \, dx = \langle f, v|_{\partial D} \rangle_{\partial D} \quad \text{for all } v \in H^1(D)/\mathbb{R},$$

where  $v|_{\partial D} := \gamma v$  and  $\gamma : H^1(D)/\mathbb{R} \rightarrow H^{1/2}(\partial D)/\mathbb{R} = (H_\diamond^{-1/2}(\partial D))'$  is the standard trace operator. Note that we occasionally write  $v$  instead of  $v|_{\partial D}$  for the sake of readability.

In his seminal paper [8], Fukushima established a one-to-one correspondence between regular symmetric Dirichlet forms and symmetric Hunt processes, which is the foundation for the construction of stochastic processes via Dirichlet form techniques. Therefore we assume that the reader is familiar with the theory of symmetric Dirichlet forms, as elaborated for instance in the monograph [10].

Let us consider the following symmetric bilinear forms on  $L^2(D)$ :

$$(5) \quad \mathcal{E}(v, w) := \int_D \kappa \nabla v(x) \cdot \nabla w(x) \, dx, \quad v, w \in \mathcal{D}(\mathcal{E}) := H^1(D)$$

and for the particular case  $\kappa \equiv 1/2$ , which is of special importance, we set

$$(6) \quad \mathcal{E}^{\text{BM}}(v, w) := \frac{1}{2} \int_D \nabla v(x) \cdot \nabla w(x) \, dx, \quad v, w \in \mathcal{D}(\mathcal{E}^{\text{BM}}) := H^1(D).$$

The pair  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  defined by (5) is a strongly local, regular symmetric Dirichlet form on  $L^2(D)$ . In particular, there exist an  $\mathcal{E}$ -exceptional set  $\mathcal{N} \subset \overline{D}$  and a conservative diffusion process  $X = (\Omega, \mathcal{F}, \{X_t, t \geq 0\}, \mathbb{P}_x)$ , starting from  $x \in \overline{D} \setminus \mathcal{N}$  such that  $X$  is associated with  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ . Without loss of generality let us assume that  $X$  is defined on the *canonical sample space*  $\Omega = C([0, \infty); \overline{D})$ . It is well-known that the symmetric Hunt process associated with (6) is the reflecting Brownian motion. Therefore, we call the symmetric Hunt process associated with (5) a *reflecting diffusion process*.

Let us briefly recall the concept of the *boundary local time* of reflecting diffusion processes, see, e.g., [4, 15, 22]. If the diffusion process is the solution to a stochastic differential equation, say the reflecting Brownian motion, then the boundary local time is given by the one-dimensional process  $L$  in the Skorohod decomposition, which prevents the sample paths from leaving  $\overline{D}$ , i.e.,

$$(7) \quad X_t = x + W_t - \frac{1}{2} \int_0^t \nu(X_s) \, dL_s,$$

$\mathbb{P}_x$ -a.s. for q.e.  $x \in \overline{D}$ . This boundary local time is a continuous non-decreasing process which increases only when  $X_t \in \partial D$ , namely for all  $t \geq 0$  and q.e.  $x \in \overline{D}$

$$L_t = \int_0^t [\partial D](X_s) \, dL_s,$$

$\mathbb{P}_x$ -a.s. and

$$\mathbb{E}_x \int_0^t [\partial D](X_s) \, ds = 0.$$

Although the reflecting diffusion process associated with (5) does in general not admit a Skorohod decomposition of the form (7), we may still define a continuous one-dimensional process with these properties. More precisely, by the Lipschitz property of  $\partial D$ , we have that  $D \cap B(x, r_D) = \{(\tilde{x}, x_d) : x_d > \gamma(\tilde{x})\} \cap B(x, r_D)$  and the Lipschitz function  $\gamma$  is differentiable a.e. with a bounded gradient. In particular, we have for every Borel set  $B \subset \partial D \cap B(x, r_D)$  that

$$\sigma(B) = \int_{\{\tilde{x} : (\tilde{x}, \gamma(\tilde{x})) \in B\}} \left(1 + |\nabla \gamma(\tilde{x})|^2\right)^{1/2} \, d\tilde{x}$$

and a straightforward computation yields that the Lebesgue surface measure  $\sigma$  is a smooth measure with respect to  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  having finite energy, i.e.,

$$\int_{\partial D} |v| \, d\sigma(x) \leq c \|v\|_{\mathcal{E}_1} \quad \text{for all } v \in \mathcal{D}(\mathcal{E}) \cap C(\overline{D}),$$

where we have used the inner product  $\mathcal{E}_1(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + \langle \cdot, \cdot \rangle$ .

**Definition 3.1.** The positive continuous additive functional of  $X$  whose Revuz measure is given by the Lebesgue surface measure  $\sigma$  on  $\partial D$ , i.e., the unique  $L \in \mathcal{A}_c^+$  such that

$$(8) \quad \lim_{t \rightarrow 0+} \frac{1}{t} \int_D \mathbb{E}_x \left\{ \int_0^t \phi(X_s) \, dL_s \right\} \psi(x) \, dx = \int_{\partial D} \phi(x) \psi(x) \, d\sigma(x)$$

for all non-negative Borel functions  $\phi$  and all  $\alpha$ -excessive functions  $\psi$ , is called the *boundary local time* of the reflecting diffusion process  $X$ .

It has been shown in the recent work [22] that the  $\mathcal{E}$ -exceptional set  $\mathcal{N}$  is actually empty.

**Proposition 1** ([22, Proposition 1]).  *$p \in C^{0,\delta}((0, T] \times \overline{D} \times \overline{D})$  for some  $\delta \in (0, 1)$ , i.e., for each fixed  $0 < t_0 \leq T$ , there exists a positive constant  $c$  such that*

$$(9) \quad |p(t_2, x_2, y_2) - p(t_1, x_1, y_1)| \leq c(\sqrt{t_2 - t_1} + |x_2 - x_1| + |y_2 - y_1|)^\delta$$

*for all  $t_0 \leq t_1 \leq t_2 \leq T$  and all  $(x_1, y_1), (x_2, y_2) \in \overline{D} \times \overline{D}$ . Moreover, the mapping  $t \mapsto p(t, \cdot, \cdot)$  is analytic from  $(0, \infty)$  to  $C^{0,\delta}(\overline{D} \times \overline{D})$ .*

By [9, Theorem 2], the existence of a Hölder continuous transition kernel density ensures that we may refine the process  $X$  to start from every  $x \in \overline{D}$  by identifying the strongly continuous semigroup  $\{T_t, t \geq 0\}$  with the transition semigroup  $\{P_t, t \geq 0\}$ . In particular, if  $v$  is continuous and locally in  $H^1(D)$ , the Fukushima decomposition holds for every  $x \in \overline{D}$ , i.e.,

$$(10) \quad v(X_t) = v(X_0) + M_t^v + N_t^v, \quad \text{for all } t > 0,$$

$\mathbb{P}_x$ -a.s., where  $M^v$  is a martingale additive functional of  $X$  having finite energy and  $N^v$  is a continuous additive functional of  $X$  having zero energy.

Moreover, both  $M^v$  and  $N^v$  can be taken to be additive functionals of  $X$  in the strict sense, cf. [10, Theorem 5.2.5].

Finally, note that the 1-potential of the Lebesgue surface measure  $\sigma$  of  $\partial D$  is the solution to an elliptic boundary value problem on a Lipschitz domain with bounded data. By elliptic regularity theory, cf., e.g., [11], this solution is continuous, implying that the boundary local time  $L$  exists as a positive continuous additive functional in the strict sense, cf. [10, Theorem 5.1.6].

For the probabilistic interpretation of the Neumann boundary value problem corresponding to the continuum model, we require the following assumption on the conductivity  $\kappa$ :

- (A1) There exists a neighborhood  $\mathcal{U}$  of the boundary  $\partial D$  such that  $\kappa|_{\mathcal{U}}$  is isotropic and equal to 1.
- (A2) There exists a neighborhood  $\mathcal{U}$  of the boundary  $\partial D$  such that  $\kappa|_{\mathcal{U}}$  is Hölder continuous.

**Remark 1.** Notice that, theoretically, the assumption (A1) imposes no restriction to generality. More precisely, it can be shown using extension techniques that for domains  $\widehat{D}, D \subset \mathbb{R}^d$  such that  $D \subset \widehat{D}$ , the knowledge of both, the Dirichlet-to-Neumann map  $\Lambda_\kappa$  on  $\partial D$  and  $\kappa|_{\widehat{D} \setminus \overline{D}}$  yields the Dirichlet-to-Neumann map  $\widehat{\Lambda}_\kappa$  on  $\partial \widehat{D}$ .

The main result for the continuum model (1), (3) is the following theorem from [22].

**Theorem 3.2** ([22, Theorem 6.4.]). *Assume that  $\kappa$  satisfies (A1) or (A2) and let  $D$  denote a bounded Lipschitz domain. Let  $f$  be a bounded Borel function satisfying  $\langle f, 1 \rangle_{\partial D} = 0$ . Then there is a unique weak solution  $u \in C(\overline{D}) \cap H_\diamond^1(D)$  to the boundary value problem (1), (3). This solution admits the Feynman-Kac representation*

$$(11) \quad u(x) = \lim_{t \rightarrow \infty} \mathbb{E}_x \int_0^t f(X_s) dL_s \quad \text{for all } x \in \overline{D}.$$

**Remark 2.** We would like to emphasize that the regularization technique employed in the proof of [22, Theorem 6.4.] may be easily modified to prove the Feynman-Kac formula

$$u(x) = \mathbb{E}_x \phi(X_{\tau(D)}), \quad x \in D$$

for the conductivity equation (1) with Dirichlet boundary condition  $u|_{\partial D} = \phi$ , where  $\phi \in H^{1/2}(D)$  and

$$\tau(D) := \inf\{t \geq 0 : X_t \in \mathbb{R}^d \setminus D\}$$

denotes the *first exit time* from the domain  $D$ . This follows from the Lipschitz property of  $\partial D$ , implying that all points of  $\partial D$  are *regular* in the sense of [18, Chapter 4.2].

#### 4. BOUNDARY PROCESSES OF REFLECTING DIFFUSIONS

**4.1. Definition and properties.** As we are going to use regularity results that are not readily available for general Lipschitz domains, we assume throughout this chapter that  $D$  has a smooth boundary in order to avoid technical difficulties. However, we expect our results to hold for general Lipschitz domains.

Let  $X^0$  denote the *absorbing diffusion process* on  $D$  which is obtained from the reflecting diffusion process  $X$  on  $\overline{D}$  by killing upon hitting of  $\mathbb{R}^d \setminus D$ , i.e.,

$$(12) \quad X_t^0 := \begin{cases} X_t, & t \leq \tau(D) \\ \partial, & t > \tau(D). \end{cases}$$

By the Markov property of  $X$ ,  $X^0$  possesses a transition kernel density which may be expressed as

$$(13) \quad p^0(t, x, y) = p(t, x, y) - \mathbb{E}_x\{p(t - \tau(D), X_{\tau(D)}, y) [\tau(D) < t]\}$$

and the regular symmetric Dirichlet form on  $L^2(D)$  associated with  $X^0$  is  $(\mathcal{E}, H_0^1(D))$ , cf. [10].

**Lemma 4.1.** *Let  $\kappa : \overline{D} \rightarrow \mathbb{R}^{d \times d}$  be a symmetric, uniformly bounded and uniformly elliptic conductivity with components  $\kappa_{ij} \in C^{0,1}(\overline{D})$ ,  $i, j = 1, \dots, d$ , such that  $\kappa$  satisfies (A1). Then the absorbing diffusion process  $X^0$  possesses a transition kernel density  $p^0$  with the following properties:*

- (i)  $p^0$  is jointly Hölder continuous with respect to  $(t, x, y)$ ;
- (ii)  $p^0$  is in  $H_0^1(D)$  as a function of  $x$  and  $y$ , respectively;
- (iii)  $p^0|_{\mathcal{U}} \in C^1(\mathcal{U} \cup \partial D) \cap C^2(\mathcal{U})$  as a function of  $x$  and  $y$ , respectively, where  $\mathcal{U}$  denotes the neighborhood of  $\partial D$  from assumption (A1);
- (iv)  $p^0$  is continuously differentiable with respect to  $t$  as a Banach space valued map.

*Proof.* The property (i) follows directly from Proposition 1 by (13).

To show the property (ii), first note that for every  $x \in D$ , we have  $v_{x,t}(y) = p^0(t/2, y, x)$  is Hölder continuous for every  $t > 0$  and in particular it is bounded and in  $L^2(D)$  since  $D$  is bounded. This implies that for every  $t > 0$  we know that

$$T_{t/2} v_x(y) = \mathcal{E}_y v_x(X_{t/2}) [t/2 < \tau_D] = \int_D p^0(t/2, y, z) p^0(t/2, y, x) dz = v_{x,2t}(y)$$

and moreover,  $T_{t/2} v_x \in \mathcal{D}(\mathcal{L}) = H_0^1(D)$ . This implies the claim.

The property (iii) is a direct consequence of assumption (A1) and the regularity of the fundamental solution for the heat equation, cf. [15].

Finally, (iv) follows by the same reasoning we used in the proof of Proposition 1, see [22].  $\square$

A function  $u \in H_{\text{loc}}^1(D)$  is said to be  $\mathcal{L}$ -harmonic, provided

$$(14) \quad \mathbb{E}_x[u(X_{\tau(\widehat{D})})] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x u(X_{\tau(\widehat{D})}) \quad \text{for all } x \in \widehat{D}$$

and every relatively compact open subset  $\widehat{D} \subset D$ , which is clearly equivalent to

$$(15) \quad \int_D \kappa \nabla u \cdot \nabla v \, dx = 0 \quad \text{for all } v \in C_c(D),$$

cf. Remark 2. By the same remark, the  $\mathcal{L}$ -harmonic extension operator  $\mathcal{H} : L^\infty(\partial D) \cap H^{1/2-\varepsilon}(\partial D) \rightarrow H_{\text{loc}}^1(D)$ ,  $\varepsilon \in [0, 1/2)$ , defined by

$$\mathcal{H}\phi(x) := \mathbb{E}_x \phi(X_{\tau(D)}), \quad x \in D$$

is well-defined. If  $\varepsilon \in (0, 1/2)$ , this is due to the fact that the Dirichlet problem

$$(16) \quad \nabla \cdot (\kappa \nabla u) = 0 \quad \text{in } D, \quad u|_{\partial D} = \phi,$$

admits a unique solution  $u \in H^{1-\varepsilon}(D) \cap H_{\text{loc}}^1(D)$  satisfying (15), which follows from a saddle point formulation introduced in [20].

The following lemma generalizes a result from Aizenman and Simon [1] for the absorbing Brownian motion.

**Lemma 4.2.** *Let  $\kappa : \overline{D} \rightarrow \mathbb{R}^{d \times d}$  be a symmetric, uniformly bounded and uniformly elliptic conductivity and suppose, in addition,  $\kappa$  satisfies (A2). Then for every bounded Borel function  $\phi$  on  $\partial D$  we have*

$$\mathbb{E}_x \phi(X_{\tau(D)}) [\tau(D) \leq t] = \int_{\partial D} \int_0^t \phi(y) \partial_{\kappa \nu(y)} p^0(s, x, y) \, ds \, d\sigma(y), \quad x \in D.$$

*Proof.* We will proceed in the spirit of the Aizenman and Simon [1] proof but we will reformulate it to cover our case. Let  $\psi \in C_c^\infty([0, \infty))$  be a smooth approximation of the indicator function  $[0 \leq s < t]$  and  $v_0(y) = p^0(\epsilon, x, y)$  for given  $x \in D$  and  $\epsilon > 0$ . Note that  $v_0 \in L^2(D)$ .

Let  $w \in H^1(D)$ . Then since  $v_s = T_s v_0$  satisfies an abstract Cauchy problem, cf., e.g., [21], we have by the equivalent variational formulation that

$$\int_0^\infty \langle \mathcal{L} v_s, w \rangle \psi(s) \, ds = \int_0^\infty \langle v_s, w \rangle \dot{\psi}(s) \, ds + \mathbb{E}_x w(X_\epsilon) [\epsilon < \tau]$$

Let  $\varphi_\epsilon$  be a non-negative  $H_0^1(D)$  function such that is an approximation of the indicator function  $[D_\epsilon]$  or more precisely  $\varphi_\epsilon[D_{2\epsilon}] = [D_{2\epsilon}]$  and  $\varphi_\epsilon[D \setminus D_\epsilon] \equiv 0$ . Rewriting  $w$  on the right-hand side by  $w = w\varphi_\epsilon + w(1 - \varphi_\epsilon)$  we get by the definition of weak derivative that

$$\begin{aligned} \langle \mathcal{L} v_s, w \rangle &= -\langle \kappa \nabla v_s, \nabla(\varphi_\epsilon w) \rangle + \langle \mathcal{L} v_s, (1 - \varphi_\epsilon)w \rangle \\ &= -\langle \kappa \nabla v_s, \nabla w \rangle + \langle \kappa \nabla v_s, \nabla((1 - \varphi_\epsilon)w) \rangle + \langle \mathcal{L} v_s, (1 - \varphi_\epsilon)w \rangle \end{aligned}$$

If  $w$  is an  $\mathcal{L}$ -harmonic, the first term on the right is 0. If  $\kappa$  is Hölder continuous close to the boundary  $\partial D$ , then the elliptic regularity shows that

$$\lim_{\epsilon \rightarrow 0} |\langle \mathcal{L} v_s, (1 - \varphi_\epsilon)w \rangle| = 0$$

since  $(1 - \varphi_\epsilon)w \rightarrow 0$  in  $H^{1-\delta}(D)$ . The term

$$\langle \kappa \nabla v_s, \nabla((1 - \varphi_\epsilon)w) \rangle = \langle \kappa \nabla v_s, (1 - \varphi_\epsilon) \nabla w \rangle - \langle \kappa \nabla v_s, w \nabla \varphi_\epsilon \rangle.$$

Since  $w \in H^1(D)$ , we have

$$\lim_{\epsilon \rightarrow 0} |\langle \kappa \nabla v_s, (1 - \varphi_\epsilon) \nabla w \rangle| = 0$$

since  $(1 - \varphi_\epsilon) \nabla w \rightarrow 0$  in  $L^2(D)$ . Since the boundary  $\partial D$  is Lipschitz and compact, it can be further divided into a finite partition of  $H^1(D)$  and within this partition the

$\nabla\varphi_\epsilon(x) = -\epsilon^{-1}[x \in D_\epsilon \setminus D_{2\epsilon}]\nu(x^*)$  where  $x^* \in \partial D$  is the unique point such that after the Lipschitz change of coordinates,  $x = \beta\nu(x^*)$ . Therefore,

$$-\lim_{\epsilon \rightarrow 0} |\langle \kappa \nabla v_s, w \nabla \varphi_\epsilon \rangle| = \int_{\partial D} \partial_{\kappa\nu(y)} v_s(y) w(y) d\sigma(y)$$

and all in all, we have shown that

$$\begin{aligned} \int_0^\infty \psi(s) \int_{\partial D} \partial_{\kappa\nu(y)} v_s(y) w(y) d\sigma(y) ds &= \int_0^\infty \langle v_s, w \rangle \dot{\psi}(s) ds + \mathbb{E}_x w(X_\epsilon) [\epsilon < \tau] \\ &\quad + o(1). \end{aligned}$$

Since the semigroup  $\{T_t, t \geq 0\}$  is continuous and  $\psi$  is an approximation of the indicator function  $[0 \leq s < t]$ , we further deduce that

$$\int_0^t \int_{\partial D} \partial_{\kappa\nu(y)} v_s(y) w(y) d\sigma(y) ds = -\langle v_t, w \rangle + w(x) + o(1)$$

for  $m$ -a.e.  $x \in D$ . Since

$$\langle v_t, w \rangle = \int_D v_t(y) w(y) dy = \int_D \int_D p^0(t, y, z) p^0(\epsilon, x, z) w(y) dy dz = T_{t+\epsilon} w(x)$$

we have deduced that

$$\int_0^t \int_{\partial D} \partial_{\kappa\nu(y)} v_s(y) w(y) d\sigma(y) ds = -T_t w(x) + w(x) + o(1)$$

for  $m$ -a.e.  $x \in D$  and for every  $\mathcal{L}$ -harmonic function  $w \in H^1(D)$ . Using the Chapman–Kolmogorov and the analyticity of  $p^0$  with respect to  $t$  we can write this as

$$\int_\epsilon^{t+\epsilon} \int_{\partial D} \partial_{\kappa\nu(y)} p^0(s, x, y) w(y) d\sigma(y) ds = -T_t w(x) + w(x) + o(1)$$

which implies that

$$(17) \quad \int_0^t \int_{\partial D} \partial_{\kappa\nu(y)} p^0(s, x, y) w(y) d\sigma(y) ds = -T_t w(x) + w(x)$$

for  $m$ -a.e.  $x \in D$ . We can use the Markov property (as in the proof of of [22, Lemma 6.3]) to show that identity (17) holds for every  $x \in D$  and for every  $\mathcal{L}$ -harmonic function  $w \in H^1(D)$ . In particular, we can reformulate this as

$$(18) \quad \begin{aligned} \int_0^t \int_{\partial D} \partial_{\kappa\nu(y)} p^0(s, x, y) \phi(y) d\sigma(y) ds &= -T_t \mathcal{H}\phi(x) + \mathcal{H}\phi(x) \\ &= -\mathbb{E}_x \mathcal{H}\phi(X_t) [t < \tau] + \mathcal{H}\phi(x) \end{aligned}$$

for every  $\phi \in H^{1/2}(\partial D)$ . By Markov property and the fact that for every  $x \in D$  the stopping time  $\tau > 0$  a.s. we have

$$\mathbb{E}_x \mathcal{H}\phi(X_t) [t < \tau] = \mathbb{E}_x \mathbb{E}(\phi(X_\tau) [t < \tau] | \mathcal{F}_t) = \mathbb{E}_x \phi(X_\tau) [t < \tau].$$

Thus, the right-hand side of the identity (18) is by the definition of the harmonic extension operator

$$-\mathbb{E}_x \phi(X_\tau) [t < \tau] + \mathbb{E}_x \phi(X_\tau) = \mathbb{E}_x \phi(X_\tau) [t \geq \tau].$$

We have therefore shown that

$$(19) \quad \int_0^t \int_{\partial D} \partial_{\kappa\nu(y)} p^0(s, x, y) \phi(y) d\sigma(y) ds = \mathbb{E}_x \phi(X_\tau) [\tau \leq t]$$



for every  $\phi \in H^{1/2}(\partial D)$ . This implies by the Monotone Class Theorem that the identity (19) holds for every bounded and measurable function  $\phi$ .  $\square$

Lemma 4.2 yields the joint distribution of the pair  $(\tau(D), X_{\tau(D)})$  with respect to the measure  $\mathbb{P}_x$ ,  $x \in \overline{D}$ , namely

$$(20) \quad \mathbb{P}_x\{\tau(D) \in dt, X_{\tau(D)} \in dy\} = \partial_{\kappa\nu(y)} p^0(t, x, y) dt d\sigma(y).$$

In particular, there exists a *Poisson kernel* with respect to the Lebesgue surface measure  $\sigma$  which is given by

$$K_\kappa(x, y) := \int_0^\infty \partial_{\kappa\nu(y)} p^0(t, x, y) dt, \quad x \in \overline{D},$$

such that the hitting probability of  $B \in \mathcal{B}(\partial D)$  for  $X$  starting in  $x \in \overline{D}$  is

$$\mathbb{P}_x\{X_{\tau(D)} \in B\} = \int_B K_\kappa(x, y) d\sigma(y).$$

Following [6], the so-called *trace Dirichlet form*  $(\widehat{\mathcal{E}}, \mathcal{D}(\widehat{\mathcal{E}}))$  of  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is given by

$$(21) \quad \widehat{\mathcal{E}}(v, w) := \mathcal{E}(\mathcal{H}v, \mathcal{H}w), \quad \mathcal{D}(\widehat{\mathcal{E}}) := \mathcal{D}_e(\mathcal{E})|_{\partial D} \cap L^2(\partial D).$$

It is a symmetric regular Dirichlet form on  $L^2(\partial D)$ , cf. [10, Theorem 6.2.1], whose domain is characterized by the following lemma.

**Lemma 4.3.** *Let  $\kappa : \overline{D} \rightarrow \mathbb{R}^{d \times d}$  be a symmetric, uniformly bounded and uniformly elliptic conductivity. Then  $\mathcal{D}(\widehat{\mathcal{E}}) = H^{1/2}(\partial D)$ .*

*Proof.* By the standard trace theorem it suffices to show  $\mathcal{D}_e(\mathcal{E}) \subset H^1(D) = \mathcal{D}(\mathcal{E})$ . Let  $v \in \mathcal{D}_e(\mathcal{E})$ . Therefore, by the definition of the extended Dirichlet space, there is a sequence  $\{v_k\}_{k \in \mathbb{N}} \subset H^1(D)$  such that  $(v_k)$  converges  $m$ -a.e. on  $D$  to  $v$  and  $(v_k)$  is  $\mathcal{E}$ -Cauchy sequence.

Since  $(v_k)$  is a  $\mathcal{E}$ -Cauchy, we have by the Poincaré inequality and the boundedness of  $\kappa$  that

$$(22) \quad \|v_n - v_m - Mv_n + Mw_m\|_{H^1(D)}^2 \leq c \|\nabla(v_n - v_m)\|_2^2 \leq c_1 \mathcal{E}(v_n - v_m, v_n - v_m),$$

where  $Mu := |D|^{-1} \langle u, 1 \rangle_2 [D]$ . In other words,  $\{v_n - Mw_n\}_{n \in \mathbb{N}}$  is an  $H^1(D)$ -Cauchy sequence and thus exists a  $w \in H^1(D)$  such that  $v_n - Mw_n \rightarrow w$  in  $H^1(D)$ . Furthermore, we may choose a subsequence  $\{w_n - Mw_n\} \subset \{v_n - Mw_n\}$  such that  $w_n - Mw_n \rightarrow w$  for  $m$ -a.e. in  $D$ . On the other hand, we have that  $w_n \rightarrow v$  for  $m$ -a.e. in  $D$  and thus,  $Mw_n = w_n - (w_n - Mw_n) \rightarrow w - v$  for  $m$ -a.e. in  $D$ . Since  $Mw_n$  is a constant function for every  $n$ , we deduce that  $w - v = c[D]$ . Therefore,  $v = w + c[D] \in H^1(D)$  as claimed.  $\square$

Now let us consider the Hunt process  $\widehat{X} = (\widehat{\Omega}, \widehat{\mathcal{F}}, \{\widehat{X}, t \geq 0\}, \widehat{\mathbb{P}}_x)$  associated with the trace Dirichlet form, which is a  $\sigma$ -symmetric pure jump process on  $\partial D$ . More precisely, as the energy measure of  $M^v$ ,  $v \in \mathcal{D}_e(\mathcal{E}) = H^1(D)$  does not charge the boundary  $\partial D$  and due to the existence of the Poisson kernel, we may apply [6, Theorem 6.2] to obtain for  $v, w \in \mathcal{D}(\widehat{\mathcal{E}}) \cap C_c(\partial D)$  the Beurling-Deny decomposition

$$(23) \quad \widehat{\mathcal{E}}(v, w) = \int_{\partial D \times \partial D \setminus \delta} (v(x) - v(y))(w(x) - w(y)) d\widehat{\mathcal{J}}(x, y).$$

The *jumping measure*  $d\widehat{\mathcal{J}}(x, y)$  is determined by the absorbing diffusion process  $X_0$  and can be characterized by the so-called *Feller measure* which depends only on the

associated symmetric strongly continuous contraction semigroup on  $L^2(D)$  and the Poisson kernel, cf. [6].  $\widehat{X}$  will be called the *boundary process* of  $X$ .

Note that the boundary process of  $X$  may be equivalently constructed using the general theory of time changes of diffusion processes with respect to positive continuous additive functionals: We have seen that the boundary local time  $L$  is a nondecreasing,  $\{\mathcal{F}_t, t \geq 0\}$ -adapted process that increases only when  $X$  is at the boundary. Following [10], we define the right-continuous right-inverse  $\tau$  of  $L$  by

$$(24) \quad \tau(s) := \sup\{r \geq 0 : L_r \leq s\}.$$

The random variable  $\tau(s)$ ,  $s \in [0, \infty)$ , is a stopping time with respect to the right-continuous history  $\{\mathcal{F}_t, t \geq 0\}$  of  $X$  since  $\{\tau(s) \geq t\} = \{L_t \leq s\} \in \mathcal{F}_t$  and moreover, by continuity of  $X$  we see that for every  $s \in [0, \infty)$  the process  $X$  is at the boundary  $\partial D$  at time  $\tau(s)$ . Therefore, we can equivalently define the boundary process  $\widehat{X}$  of  $X$  as the time-changed trace

$$\widehat{X}_t := X_{\tau(t)}$$

and the *boundary filtration*

$$\widehat{\mathcal{F}}_t := \mathcal{F}_{\tau(t)}.$$

To be precise, we know that the boundary local time  $L$  is a positive continuous additive functional in the strict sense since the corresponding 1-potential is bounded by elliptic regularity and therefore, the boundary  $\partial D$  coincides with the so-called *quasi-support* of  $L$ , cf. [10, Theorem 5.1.5]. Moreover, since  $\partial D$  is smooth, every boundary point is a *regular* point, cf. [18], so that the boundary process is a  $\sigma$ -symmetric Hunt process on the boundary  $\partial D$ , cf. [10, Theorem A.2.12., Theorem 6.2.1].

**Remark 3.** Note that due to the refinement obtained by Proposition 1 for the reflecting diffusion process  $X$ , the boundary process  $\widehat{X}$  can be defined without exceptional set, i.e., for every starting point  $x \in \partial D$ .

We note that the representation Theorem 3.2 can be expressed with the help of the boundary process  $\widehat{X}$ , the first exit time  $\tau_D$  and the first exit place  $X_{\tau_D}$ .

**Lemma 4.4.** *We have that*

$$(25) \quad \int_{\tau(a)}^{\tau(b)} f(s) dL_s = \int_a^b f(\tau(s)) ds$$

for every bounded and measurable function  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

*Proof.* This follows by Monotone Class Theorem from the observation that

$$[L_a, L_b] = \tau \circ g_{a,b},$$

where we have set  $g_{a,b}(t) := [t \in [a, b]]$ . □

**4.2. Infinitesimal generator.** The following theorem yields the infinitesimal generator of the boundary process.

**Theorem 4.5.** *Let  $\kappa: \overline{D} \rightarrow \mathbb{R}^{d \times d}$  be a symmetric, uniformly bounded and uniformly elliptic conductivity. Then the infinitesimal generator of  $\widehat{X}$  is the Dirichlet-to-Neumann map  $\Lambda_\kappa$ .*

*Proof.* We will give two different proofs for this result. One is with the trace Dirichlet forms and one with the help of Theorem 3.2 and Change of Variables lemma 4.4.

Let  $v, w \in \mathcal{D}(\widehat{\mathcal{E}}) = H^{1/2}(\partial D)$ . Therefore,

$$\begin{aligned}\widehat{\mathcal{E}}(v, w) &= \mathcal{E}(\mathcal{H}v, \mathcal{H}w) = \int_{\partial D} w(x) (\partial_{\kappa\nu(x)} \mathcal{H}v(x))|_{\partial D} d\sigma(x) \\ &= \int_{\partial D} w(x) (\partial_{\kappa\nu(x)} \mathcal{H}v(x))|_{\partial D} d\sigma(x) \\ &= \langle w, \Lambda_\kappa v \rangle_{H^{1/2}(\partial D), H^{-1/2}(\partial D)}\end{aligned}$$

We can factorize  $\Lambda_\kappa = A^*A$ , where  $A: H^{1/2}(\partial D) \rightarrow L^2(\partial D)$ . The  $A$  can be seen as an unbounded operator in  $L^2(\partial D)$  with domain  $\mathcal{D}(\widehat{\mathcal{E}})$ . Thus,

$$\widehat{\mathcal{E}}(v, w) = \langle Aw, Av \rangle_{L^2(\partial D)},$$

which implies  $A = \sqrt{-\mathcal{L}}$  or  $\Lambda_\kappa = -\mathcal{L}$ , where  $\mathcal{L}$  is the infinitesimal generator of  $\widehat{X}$  (c.f. [10]).

The another proof is even more probabilistic in nature. Let  $\varphi \in C^\infty(\partial D)$  and  $v = \Lambda_\kappa \varphi$ . We know that  $v$  is Hölder continuous and  $\langle v, 1 \rangle_{\partial D} = 0$ . Therefore, we may define

$$g_R(x) := \mathbb{E}_x A_R := \mathbb{E}_x \int_0^R v(\widehat{X}_s) ds \quad \text{and} \quad g_\infty(x) = \lim_{R \rightarrow \infty} g_R(x).$$

Note that the Theorem 3.2 together with the Change of Variables Lemma 4.4 gives  $g_\infty = \Lambda_\kappa^{-1}v = \varphi$  and in particular,  $g_\infty \in C^\infty(\partial D) \subset \mathcal{D}(\mathcal{L})$ . The transition operator  $\widehat{T}_t$  applied to  $g$  gives by Markov property

$$\widehat{T}_t g_R(x) = \mathbb{E}_x \mathbb{E}_{\widehat{X}_t} A_R = \mathbb{E}_x (A_{R+t} - A_t).$$

Therefore,

$$(\widehat{T}_t - I)g_R(x) = -\mathbb{E}_x A_t + \mathbb{E}_x (A_{R+t} - A_R)$$

and the Theorem 3.2 together with dominated convergence theorem imply that

$$(\widehat{T}_t - I)\varphi(x) = -\mathbb{E}_x A_t = -\int_0^t \widehat{T}_s v(x) ds.$$

This in turn implies

$$\mathcal{L}\varphi(x) = -v(x) = -\Lambda_\kappa \varphi(x)$$

which implies the claim since  $(\widehat{\mathcal{E}}, \mathcal{D}(\widehat{\mathcal{E}}))$  is regular.  $\square$

As in [17], we show that the Dirichlet-to-Neumann map is an *integro-differential operator* in the sense of Lepeltier and Marchal [19].

**Theorem 4.6.** *Let  $\kappa: \overline{D} \rightarrow \mathbb{R}^{d \times d}$  be a symmetric, uniformly bounded and uniformly elliptic conductivity with components  $\kappa_{ij} \in C^{0,1}(\overline{D})$ ,  $i, j = 1, \dots, d$  such that  $\kappa$  satisfies (A1). Then the Dirichlet-to-Neumann map  $\Lambda_\kappa$  is of form*

$$\Lambda_\kappa \phi = \Lambda_\kappa \text{id} \cdot \nabla_T \phi + A_\kappa \phi \quad \text{for all } \phi \in H^{1/2}(\partial D),$$

where  $A_\kappa$  is given by the integral operator

$$A_\kappa \phi(x) := \int_{\partial D} (\phi(y) - \phi(x) - \nabla_T \phi(x) \cdot (y - x)) N_\kappa(x, y) d\sigma(y)$$

for a.e.  $x \in \partial D$  and

$$N_\kappa(x, y) := \partial_{\kappa\nu(x)} K_\kappa(x, y), \quad x, y \in \partial D.$$

*Proof.* By a density argument, we may assume that  $\phi \in C^\infty(\partial D) \cap H^{1/2}(\partial D)$ . The unique solution in  $H^1(D)$  of the Dirichlet problem (16) is by Lemma 4.2

$$\mathcal{H}\phi(x) = \mathbb{E}_x \phi(X_{\tau_D}) = \int_{\partial D} \int_0^\infty \phi(y) \partial_{\kappa\nu(y)} p^0(t, x, y) dt d\sigma(y).$$

Therefore,  $\Lambda_\kappa$  maps  $\phi$  to

$$(26) \quad \Lambda_\kappa \phi(x) = \partial_{\kappa\nu(x)} \mathcal{H}\phi(x).$$

Let us extend  $\phi$  into the neighborhood of the boundary as constant along the conormal direction. Therefore, the first order tangential derivative  $V := \nabla_T \phi$  gets extended simultaneously. We will denote the extensions  $\tilde{\phi}$  and  $\tilde{V}$ , respectively. We compute the conormal derivative of the function

$$w := \mathcal{H}\phi - \tilde{\phi} \mathcal{H}1 - \sum_{j=1}^d \tilde{V}_j (\mathcal{H}W_j - \tilde{W}_j \mathcal{H}1),$$

where  $\{W\}$  is a vector field on the boundary defined by  $W(y) := y_T$  as the projection to the tangent plane going through the point  $y$ . By construction, the conormal derivative commutes with multiplication by the extended functions and vector fields. Therefore,

$$\partial_{\kappa\nu} w = \Lambda_\kappa \phi - \phi \Lambda_\kappa 1 - \sum_{j=1}^d V_j \Lambda_\kappa W_j = \Lambda_\kappa \phi - V \cdot \Lambda_\kappa W,$$

where  $\Lambda_\kappa 1 = 0$  since  $u(x) = 1$  in  $\overline{D}$  is the unique solution to the corresponding Dirichlet problem and therefore, the conormal derivative vanishes identically. As we have shown the existence of a Poisson kernel, we can write the left-hand side in a different way, namely

$$\nabla w(x) = \nabla_x \int_{\partial D} (\phi(y) - \tilde{\phi}(x) - \tilde{V}(x) \cdot (\tilde{W}(y) - \tilde{W}(x))) K_\kappa(x, y) d\sigma(y)$$

for almost every  $x$  in a neighborhood of the boundary.

Next, we show that the function  $y \mapsto N_\kappa(x, y)(|y - x|^2 \wedge 1)$  is integrable with respect to the surface measure  $\sigma$  for every  $x \in \partial D$ . When  $\kappa \equiv 1$ , the proof given in [17] yields the claim for the kernel  $N_1$  corresponding to  $\kappa \equiv 1$ . As we assume that  $\kappa$  satisfies (A1), the operator  $\Lambda_\kappa - \Lambda_1$  is a smoothing operator which follows by the standard elliptic regularity, cf. [14]. This implies that the kernels  $N_1$  and  $N$  have the same leading singularities and the claim thus follows from the estimate for  $N_1$ . As a consequence, we can use the dominated convergence theorem to take the differentiation inside the integration and we obtain

$$\partial_{\kappa\nu} w(x) = \int_{\partial D} (\phi(y) - \phi(x) - \nabla_T \phi(x) \cdot (y - x)) N_\kappa(x, y) d\sigma(y) = A_\kappa \phi(x)$$

for a.e.  $x \in \partial D$ . □

## 5. NOVEL PROBABILISTIC FORMULATIONS OF THE CALDERÓN PROBLEM

By the considerations from above, the boundary process is uniquely determined by the absorbing diffusion process  $X^0$  so that Theorem 4.5 leads to the following probabilistic interpretation of Calderón's problem: *Given a boundary process  $\hat{X}$  associated with the regular  $\sigma$ -symmetric Dirichlet form  $(\hat{\mathcal{E}}, H^{1/2}(\partial D))$ , is  $X^0$  the unique absorbing diffusion process on  $D$  such that  $\hat{X}$  is the boundary process of the corresponding reflecting diffusion process  $X$  on  $\overline{D}$ ?*

The Calderón problem in 2 dimensions is known to be solvable for isotropic  $\kappa \in L^\infty(D)$ . Given the boundary process  $\hat{X} = \hat{X}_{\kappa_0}$  we can thus uniquely determine the generator  $\Lambda = \Lambda_{\kappa_0}$ . The celebrated result of Astala and Päivärinta [3] says that whenever  $\Lambda_\kappa = \Lambda$  and both  $\kappa$  and  $\kappa_0$  are isotropic, uniformly bounded and uniformly elliptic, then  $\kappa = \kappa_0$ . Therefore, the equality  $X_\kappa = X_{\kappa_0}$  must hold as well.

The recent result by Haberman and Tataru [13] implies the same for three and higher when  $\kappa$  and  $\kappa_0$  are assumed to be  $C^1(D)$  or if they are Lipschitz continuous and close to identity in certain sense. In recent preprints, Haberman [12] improved this to the even  $W^{1,n}$  conductivities for the dimensions three and four, while Caro and Rogers [5] improved the Haberman and Tataru technique to prove the uniqueness for the Lipschitz case in general in any dimension.

When the conductivity is not assumed to be isotropic the uniqueness has always an obstruction, namely we have  $\Lambda_{\kappa_0} = \Lambda_{\kappa_1}$ , whenever  $\kappa_1 = F_*\kappa_0$  is the push-forward conductivity by a diffeomorphism  $F$  on  $D$  that leaves the boundary  $\partial D$  invariant. In the plane, this is known to be the only obstruction by the result of Astala, Lassas and Päivärinta [2] which holds without additional regularity assumptions on the conductivity. In higher dimensions the question is still very much open in general, see [7] for further discussion.

It should be emphasized that these results from analysis all rely on so-called *complex geometric optics solutions* and the authors are not aware of any probabilistic interpretation of such solutions. Therefore, it is an open problem to find a probabilistic solution to the probabilistic formulation of Calderón's problem.

Let us elaborate a bit on this problem by providing three seemingly different but equivalent versions.

**5.1. First version.** It is an immediate consequence of Theorem 4.6 that the jumping measure of  $\hat{X}$  can be described with the help of [19, Théorème 10].

**Lemma 5.1.** *Let  $\kappa : \overline{D} \rightarrow \mathbb{R}^{d \times d}$  be a symmetric, uniformly bounded and uniformly elliptic conductivity with components  $\kappa_{ij} \in C^{0,1}(\overline{D})$ ,  $i, j = 1, \dots, d$ , such that  $\kappa$  satisfies (A1). Then for every non-negative Borel function  $\phi : \partial D \times \partial D \rightarrow \mathbb{R}_+$ , vanishing on the diagonal, and any stopping time  $\hat{\tau}$  of  $\hat{X}$ , we have*

$$(27) \quad \mathbb{E}_x \sum_{s \leq \hat{\tau}} \phi(\hat{X}_{s-}, \hat{X}_s) [\hat{X}_{s-} \neq \hat{X}_s] = \mathbb{E}_x \int_0^{\hat{\tau}} \int_{\partial D} \phi(\hat{X}_s, y) N_\kappa(\hat{X}_s, y) d\sigma(y) ds.$$

*Proof.* Suppose first that  $\text{diam}(D) \leq 1$ . We note that the operator  $A_\kappa$  in Theorem 4.6 coincides with the integral operator causing the jumps. If  $\psi \in C^\infty(\partial D) \cap H^{1/2}(\partial D)$  and it is continued as  $\tilde{\psi} \in H^2(\mathbb{R}^d)$  so that  $\psi$  and its tangential derivative are continued as constants along the conormal directions in the neighborhood of the boundary  $\partial D$ , then for every  $x \in \partial D$ , we have

$$A_\kappa \psi(x) = \int_{\mathbb{R}^d \setminus \{0\}} (\tilde{\psi}(x+z) - \tilde{\psi}(x) - [|z| \leq 1] z \cdot \nabla \tilde{\psi}(x)) S_\kappa(x, dz),$$

where we have set  $S_\kappa(x, dz) := N_\kappa(x, x+z) d\sigma_x(z)$  and  $\sigma_x(B) := \sigma(x+B)$  for every Borel set  $B \subset \mathbb{R}^d$ . In the same way we can extend the drift term so that the corresponding integro-differential operator is the infinitesimal generator of an extended process  $\overline{X}$  on the whole space  $\mathbb{R}^d$ . Since we know that  $\hat{X}_t \in \partial D$  for all  $t \geq 0$ , it follows that the extended process  $\overline{X}$  will stay on the boundary  $\partial D$  if we start it from the boundary and it coincides with  $\hat{X}$  there.

For this extended process  $\overline{X}$  we can apply the result [19, Théorème 10] and we obtain

$$\begin{aligned} & \mathbb{E}_x \sum_{s \leq \hat{\tau}} \phi(\hat{X}_{s-}, \hat{X}_s) [\hat{X}_{s-} \neq \hat{X}_s] \\ &= \mathbb{E}_x \int_0^{\hat{\tau}} \int_{\mathbb{R}^d \setminus \{0\}} \phi(\hat{X}_s, \hat{X}_s + y) N_\kappa(\hat{X}_s, \hat{X}_s + y) d\sigma_{\hat{X}_s}(y) ds \end{aligned}$$

for any non-negative Borel function  $\phi : \partial D \times \partial D \rightarrow \mathbb{R}_+$  vanishing on the diagonal and any stopping time  $\hat{\tau}$  of  $\hat{X}$ . The claim follows now in this special case by change of integration variable.

The general case follows by scaling: Let us denote  $X_t^R := R^{-1}X_t$ . This is a reflecting diffusion process corresponding to  $\kappa^R$  on a domain  $D^R$ , where  $D^R := R^{-1}D$  and  $\kappa^R(x) := R^{-2}\kappa(Rx)$ . Since the diameter of  $D^R$  is one, the claim holds for the boundary process  $\hat{X}^R$  of  $X^R$ .

Let  $L^R$  denote the local time of  $X^R$  on the boundary  $\partial D^R$ . By definition, this is in Revuz correspondence with the surface measure  $\sigma^R$  of the boundary  $\partial D^R$ . By using the Revuz correspondence and change of variables, it follows that

$$L_t^R = RL_t.$$

Therefore, the right-inverse  $\tau^R$  of the local time  $L^R$  has a scaling law

$$\tau^R(t) = \tau(R^{-1}t)$$

which in turn implies that the boundary processes scale by the law

$$\hat{X}_t^R = R^{-1}\hat{X}_{R^{-1}t}$$

and that  $\eta$  is an  $\hat{X}$ -stopping time if and only if  $R\eta$  is an  $\hat{X}^R$ -stopping time.

If we denote by  $N_\kappa^R$  the conormal derivative of the Poisson kernel of  $X^R$  multiplied by 2 and compute the scaling law, we find out that

$$N_\kappa^R(x, y) = R^{d-2}N_\kappa(Rx, Ry).$$

With all these scaling laws, we are now ready to prove the claim for  $\hat{X}$ . Let  $\phi^R(x, y) := \phi(x, y)$  and let  $\eta$  be an  $\hat{X}$ -stopping time. We have

$$\mathbb{E}_x \sum_{s \leq \eta} \phi(\hat{X}_s, \hat{X}_{s-}) [\hat{X}_s \neq \hat{X}_{s-}] = \widehat{\mathbb{E}}_{R^{-1}x} \sum_{s \leq \eta R} \phi^R(\hat{X}_s^R, \hat{X}_{s-}^R) [\hat{X}_s^R \neq \hat{X}_{s-}^R],$$

where  $\widehat{\mathbb{E}}_{R^{-1}x}$  denotes the expectation given  $\hat{X}_0^R = R^{-1}x$ . By the first part of the proof, the right-hand side is equal to

$$\widehat{\mathbb{E}}_{R^{-1}x} \int_0^{\eta R} \int_{\partial D^R} \phi^R(\hat{X}_s^R, y) N_\kappa^R(\hat{X}_s^R, y) d\sigma(y) ds.$$

With the change of variables  $y' = Ry$  and  $s' = sR^{-1}$  and the scaling law  $N_\kappa^R(\hat{X}_{Rs}^R, R^{-1}y) = R^{d-2}N_\kappa(\hat{X}_s, y)$ , the claim follows.  $\square$

This result states that the pair  $(N_\kappa(x, y) d\sigma(y), \text{id}_t)$  is a *Lévy system* of the Hunt process  $\hat{X}$ . Since the positive continuous additive functional  $\text{id}_t = t$  with respect to  $\hat{X}$  has the Revuz measure  $\sigma$ , we obtain from [6, Theorem A.1 (iii)] that  $\frac{1}{2}N_\kappa(x, y) d\sigma(y) d\sigma(x)$  coincides with the jumping measure  $d\hat{\mathcal{J}}(x, y)$  of the boundary process  $\hat{X}$ . In particular, the Lévy system characterizes the boundary process completely and we have obtained another probabilistic formulation of Calderón's problem: *Given pure jump processes on*

$\partial D$  generated by  $\Lambda_{\kappa_1}$  and  $\Lambda_{\kappa_2}$ , respectively. Show that the corresponding Lévy systems coincide, i.e.,  $N_{\kappa_1} \equiv N_{\kappa_2}$ , if and only if  $\kappa_1 \equiv \kappa_2$ .

**5.2. Second version.** An interesting aspect with regard to the first version is the fact that it may be translated into an equivalent parabolic *unique continuation problem* in terms of transition kernel densities of absorbing diffusion processes on  $D$ . More precisely, let  $\kappa_1, \kappa_2 \in C^{0,1}(\overline{D})$  be isotropic conductivities and  $p_1^0, p_2^0$  the corresponding transition kernel densities. Define for any function  $\phi \in L^2(D)$  the functions

$$v(t, x) := \int_D (p_1^0(t, x, y) - p_2^0(t, x, y)) \phi(y) dy, \quad w(t, x) := \int_D p_2^0(t, x, y) \phi(y) dy$$

and set

$$c(t, x) := -\nabla \cdot (\kappa_1 \nabla w(t, x)) + \nabla \cdot (\kappa_2 \nabla w(t, x)).$$

By these definitions, we have

$$(28) \quad \begin{cases} \partial_t v(t, x) = \nabla \cdot (\kappa_1 \nabla v(t, x)) + c(t, x), & (t, x) \in (0, \infty) \times D \\ v(0, x) = 0, & x \in D \\ v(t, x) = 0, & (t, x) \in (0, \infty) \times \partial D. \end{cases}$$

By the probabilistic formulation from above, the uniqueness result from Caro and Rogers [5] corresponds to the following assertion: *Assume that*

$$(29) \quad \int_0^\infty \partial_{\kappa\nu(x)} \partial_{\kappa\nu(y)} (p_1^0(t, x, y) - p_2^0(t, x, y)) dt \equiv 0.$$

*Then  $v$  must be identically zero for any  $\phi \in L^2(D)$ .*

**5.3. Third version.** Let us conclude this section by deriving yet another equivalent formulation of Calderón's problem which is based on the construction for the reflecting Brownian motion from [17]. To be precise, as in [17], we adopt Itô's idea of regarding the excursions of  $X$  from the boundary as a point process taking values in the space of excursions. First, we recall some definitions from [17]. A measurable function  $e : [0, \infty) \rightarrow \partial D \cup \{\partial\}$  is called a *point function* if the set  $J(e) := \{s > 0 : e_s \in \partial D\}$  is countable and we denote the set of point functions by  $\mathcal{P}(\partial D)$ . For each point function  $e$ , the *counting measure* on  $(0, \infty) \times \partial D$  is defined by

$$(30) \quad n_e(B) := \{(s, x) \in B : e_s = x\}, \quad B \subset (0, \infty) \times \partial D.$$

Given the probability space  $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}_x)$  with filtration  $\{\widehat{\mathcal{F}}_t, t \geq 0\}$ , a function  $e : \widehat{\Omega} \rightarrow \mathcal{P}(\partial D)$  is called a *point process* if for each  $B \subset \mathcal{B}(\partial D)$ , the increasing process  $t \mapsto n_e((0, t] \times B)$  is adapted to  $\{\widehat{\mathcal{F}}_t, t \geq 0\}$ .

Let  $W^{a,b}$  denote the space of continuous *excursions* of  $X$  from  $a \in \partial D$  to  $b \in \partial D$ , i.e., the space of continuous paths  $e \in C([0, \infty); \overline{D})$  such that  $e(0) = a$  and there exists an  $l > 0$  such that  $e(t) \in D$  for all  $0 < t < l$  and  $e(t) = b$  for all  $t \geq l$ . If the space  $W^{a,b}$  is equipped with the filtration  $\sigma(e(s), e \in W^{a,b}, 0 \leq s \leq t)$ , then the space of all excursions is given by

$$(31) \quad W = \bigcup_{a \in \partial D} W^a,$$

where

$$(32) \quad W^a := \bigcup_{b \in \partial D, b \neq a} W^{a,b}.$$

The same proof as in [17, Proposition 4.4] shows that the random set of jump times  $\{\widehat{X}_{s-} \neq \widehat{X}_s\}$  is a countable and dense set and that there is a constant  $c > 0$ , depending only on the domain  $D$ , such that after any given time  $t \geq 0$ , there are always infinitely many jumps of size at least  $c$ . Therefore, we set

$$J := \{s \in (0, \infty) : \tau(s-) < \tau(s)\} \quad \text{and} \quad l(s) := \tau(s) - \tau(s-)$$

and define the *point process of excursions* of  $X$  as a  $W$ -valued point process such that

$$(33) \quad e_s(t) := \begin{cases} X_{t+\tau(s-)}, & t \leq l(s) \\ X_{\tau(s)}, & t > l(s), \end{cases}$$

if  $s \in J$ , whereas  $e_s := \partial$  if  $s \notin J$ .

Clearly, the family  $\{e_t, t \leq 0\}$  is adapted to  $\{\widehat{\mathcal{F}}_t, t \geq 0\}$ . The following definition yields a useful characterization of point processes.

**Definition 5.2.** A  $\sigma$ -finite random measure  $\hat{n}_e$  on the measurable space  $((0, \infty) \times \partial D, \mathcal{B}((0, \infty)) \times \mathcal{B}(\partial D))$  is called the *compensating measure* of a point process  $e$  if there is a sequence  $\{B_k\}_{k \in \mathbb{N}} \subset \mathcal{B}(\partial D)$ , exhausting  $\partial D$ , such that

- (i) the mapping  $t \mapsto \hat{n}_e((0, t) \times B_k)$  is continuous for every  $k \in \mathbb{N}$ ;
- (ii)  $\mathbb{E} \hat{n}_e((0, t) \times B_k) < \infty$  for every  $k \in \mathbb{N}$ ;
- (iii) for every  $B \in \mathcal{B}(\partial D)$  contained in some  $B_k$ , the process

$$t \mapsto n_e((0, t) \times B) - \hat{n}_e((0, t) \times B)$$

is a martingale with respect to  $\{\widehat{\mathcal{F}}_t, t \geq 0\}$ .

The following theorem is a straightforward generalization of [15, Theorem 5.1] for the reflecting Brownian motion. It provides a *compatibility condition*, connecting the excursion law  $\mathbb{P}_{a,b}$  on  $W^{a,b}$ , which is completely determined by  $p^0$ , with the Lévy system of  $\widehat{X}$ .

**Theorem 5.3.** Let  $\kappa : \overline{D} \rightarrow \mathbb{R}^{d \times d}$  be a symmetric, uniformly bounded and uniformly elliptic conductivity with components  $\kappa_{ij} \in C^{0,1}(\overline{D})$ ,  $i, j = 1, \dots, d$ , such that  $\kappa$  satisfies (A1). Then the point process of excursions of  $X$  admits the compensating measure

$$(34) \quad \hat{n}_e((0, t) \times B) = \int_0^t \mathbb{Q}_{\widehat{X}_s} \{B \cap \{e : e(0) = \widehat{X}_s\}\} ds,$$

where the excursion law  $\mathbb{Q}_a$  from  $a \in \partial D$  is a  $\sigma$ -finite measure on  $W^a$  defined by

$$(35) \quad \mathbb{Q}_a\{B\} = \int_{\partial D} \mathbb{P}_{a,b}\{B \cap \{e : e(l) = b\}\} N_\kappa(a, b) d\sigma(b)$$

for every measurable  $B \subset W^a$ .

We have thus arrived at the following equivalent probabilistic formulation of Calderón's problem: *Given the boundary process  $\widehat{X}$  with its corresponding Lévy system  $(N_\kappa(x, y) d\sigma(y), \text{id}_t)$ , show that  $\mathbb{P}_{a,b}$  determined by  $p^0$  is the unique excursion law such that (34) yields the compensating measure of the point process of excursions of some reflecting diffusion process on  $\overline{D}$ .*



## 6. CONCLUSION AND OUTLOOK

In this work, we have obtained a probabilistic formulation of Calderón's inverse conductivity problem. This formulation comes in three equivalent versions and each of them may yield both a novel perspective as well as a novel set of (probabilistic) tools when it comes to studying questions related to the unique determinability of conductivities from boundary data. Indeed, it was shown in [17] that for the case  $\kappa \equiv 1/2$  and under a certain consistency assumption, the reflecting Brownian motion can be reconstructed from its point process of excursions and the boundary process. However, the consistency assumption was derived by using the reflecting Brownian motion to begin with and as it is noted in [17], there might be other consistency assumptions leading to other constructions. Showing that there are no other consistent constructions is equivalent to the probabilistic inverse problem which will be the subject of future research.

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